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THERMODYNAMIC INFLUENCES ON ONE DIMENSIONAL SHOCK WAVES AND INDUCED DISCONTINUITIES IN THERMOELASTIC BODIES AND SECOND ORDER EFFECTS

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Abstract—The behavior of one dimensional shock waves and induced discontinuities propagating in thermoelastic bodies is examined. We consider specific representations of the constitutive relations and show that to within second order of the shock strength differences between the predictions of thermoelastic and elastic theories are manifested. In particular, the shock speed is greater, the evolutionary behavior of the shock amplitude and that of the induced discontinuity are less pronounced in the thermoelastic theory. We also exhibit certain features of the solution of the coupled equations governing the behavior of shocks and induced discontinuities.

1. INTRODUCTION

The question of thermodynamic effects on the behavior of mechanical waves in material bodies has been of long-standing interest to researchers in the field. Although general results have been obtained concerning the evolutionary behavior of shock and acceleration waves including considerations of thermodynamic influences, detailed comparisons with the results based on purely mechanical theories have been somewhat difficult. This is due mainly to the general nature of the constitutive functions which are not specified explicitly. It would therefore be of some interest to examine particular representations of the constitutive functions whereby certain detailed comparisons of the results may be made.

In this paper we derive the governing differential equation of induced discontinuities behind one dimensional shock waves in non-heat-conducting thermoelastic bodies, and consider approximations of this equation and the governing equation of shock amplitudes. It is shown that to within terms of first order of the shock strength, there is no difference between the predictions of thermodynamic and purely mechanical theories. However, to within terms of second order of the shock strength, the speed of propagation of the shocks and the evolutionary behavior of the shocks and the induced discontinuities depend on the thermodynamic properties of the materials. These latter results are somewhat surprising in view of the fact that the jump in entropy across a shock is proportional to third order of the shock strength in the limit as the shock becomes weak.

2. BASIC EQUATIONS

In one dimension the constitutive relations for the internal energy e, the stress T and the absolute temperature θ of thermoelastic bodies are given by

$$\begin{aligned} e &= \hat{e}(S,\eta), \\ T &= \hat{T}(S,\eta), \\ \theta &= \hat{\theta}(S,\eta), \end{aligned} \tag{2.1}$$

where S is the strain and η is the entropy. The strain S is given by

$$S = u_X, \tag{2.2}$$

where $u(X,t)$ is the displacement at time t of the material point X identified with its position in a fixed homogeneous reference configuration with mass density ρ_R. The constitutive response functions are not independent, i.e.

$$\hat{T} = \hat{e}_S, \qquad \hat{\theta} = \hat{e}_\eta. \tag{2.3}$$

We define the functions

$$E = \hat{T}_S, \qquad \tilde{E} = \hat{T}_{SS}, \qquad G = \hat{T}_\eta, \tag{2.4}$$

and we require that

$$E(S,\eta) > 0, \qquad \tilde{E}(S,\eta) < 0, \qquad G(S,\eta) < 0. \tag{2.5}$$

The second inequality ensures that a compression shock may exist, and the third inequality holds for most materials.

In the absence of a singular surface, balance of linear momentum and balance of energy imply that

$$\begin{aligned} T_X &= \rho_R \ddot{u}, \\ \dot{e} &= T\dot{S}. \end{aligned} \tag{2.6}$$

The balance equations and the constitutive relations yield a system of equations for the determination of u and η.

A shock wave is a propagating singular surface across which the displacement is continuous, but its derivations suffer jump discontinuities. The shock speed is defined by

$$V(t) = \frac{\mathrm{d}}{\mathrm{d}t} Y(t), \tag{2.7}$$

where $X = Y(t)$ is the material point at which the wave is to be found at time t. There is no loss in generality in assuming that $V(t) > 0$. Balance of linear momentum and balance of energy imply that across a shock wave

$$\begin{aligned} [T] &= -\rho_R V[\dot{u}], \\ -V[e+\tfrac{1}{2}\rho_R \dot{u}^2] &= [T\dot{u}]. \end{aligned} \tag{2.8}$$

Here, for any function f, $[f] = f^- - f^+$ with $f^\mp = \lim_{X \to Y(t)^\mp} f(X,t)$.

Central to the study of wave propagation in one dimension is the kinematical condition of compatibility

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = [\dot{f}] + V[f_X], \tag{2.9}$$

where $\mathrm{d}/\mathrm{d}t$ is called the displacement derivative. For a shock wave with $f = u$, formula (2.9) implies

$$[\dot{u}] = -V[S]. \tag{2.10}$$

Formulae $(2.8)_1$ and (2.10) yield the result for the shock speed, viz.

$$\rho_R V^2 = \frac{[T]}{[S]}. \tag{2.11}$$

Further, $(2.8)_2$ may be rewritten in the form

$$[e] = \tfrac{1}{2}(T^- + T^+)[S]. \tag{2.12}$$

An interesting implication of balance of energy $(2.6)_2$ and (2.3) is that

$$\theta\dot{\eta} = 0. \tag{2.13}$$

Since $\theta(X, t) > 0$, we conclude that $\dot{\eta}(X, t) = 0$.

The governing differential equation of the shock amplitude $[S]$ has been derived previously by Chen and Gurtin[1]. They showed that for a shock wave propagating in a thermoelastic body which is initially at rest in its reference configuration the governing differential equation is of the form

$$\frac{d}{dt}[S] = Vg[S_X], \tag{2.14}$$

where

$$g = -\frac{(E^- - \rho_R V^2)(2\theta^- - G^-[S])}{(3\rho_R V^2 + E^-)\theta^- - (3\rho_R V^2 - E^-)G^-[S]}. \tag{2.15}$$

The implications of (2.14) on the evolutionary behavior of the shock amplitude have also been deduced previously. We shall appeal to some of the results later on in this paper.

3. DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATION OF INDUCED DISCONTINUITIES

To begin with let us derive certain implications of the kinematical condition of compatibility (2.9). First of all, we note that

$$\frac{d}{dt}[f] = [\dot{f}] + V[f_X],$$

$$\frac{d}{dt}[f_X] = [\dot{f}_X] + V[f_{XX}],$$

so that

$$\frac{d}{dt}[\dot{f}] = [\ddot{f}] + V\frac{d}{dt}[f_X] - V^2[f_{XX}]. \tag{3.1}$$

Next, the derivative of (2.9) yields

$$\frac{d}{dt}[\dot{f}] = \frac{d^2}{dt^2}[f] - V\frac{d}{dt}[f_X] - \frac{dV}{dt}[f_X]. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain the desired result

$$[\ddot{f}] = V^2[f_{XX}] - 2V\frac{d}{dt}[f_X] - \frac{dV}{dt}[f_X] + \frac{d^2}{dt^2}[f], \tag{3.3}$$

which we shall use later on.

The derivative of the equation of balance of linear momentum $(2.6)_1$ and the constitutive relation for the stress $(2.1)_2$ yield

$$\bar{E}(S_X)^2+E(S_{XX})+2\hat{T}_{S\eta}S_X\eta_X+\hat{T}_{\eta\eta}(\eta_X)^2+G(\eta_{XX}) = \rho_R\ddot{S},$$

so that its jump across a shock wave in a material body which is initially at rest in its reference configuration is

$$\bar{E}^-[S_X]^2+E^-[S_{XX}]+2\hat{T}^-_{S\eta}[S_X][\eta_X]+\hat{T}^-_{\eta\eta}[\eta_X]^2+G^-[\eta_{XX}] = \rho_R[\ddot{S}]. \tag{3.4}$$

Formula (3.3) with $f = S$ becomes

$$[\ddot{S}] = V^2[S_{XX}]-2V\frac{\mathrm{d}}{\mathrm{d}t}[S_X]-\frac{\mathrm{d}V}{\mathrm{d}t}[S_X]+\frac{\mathrm{d}^2}{\mathrm{d}t^2}[S]. \tag{3.5}$$

The governing differential equation of induced discontinuities follows from the evaluation of (3.4) and (3.5). Before this can be done we need to derive certain additional results.

First, we recall that†

$$\frac{\mathrm{d}V}{\mathrm{d}t} = VI\frac{\mathrm{d}}{\mathrm{d}t}[S], \tag{3.6}$$

where

$$I = \frac{(E^- -\rho_R V^2)\theta^-}{\rho_R V^2(2\theta^- - G^-[S])[S]} \tag{3.7}$$

and

$$[\eta_X] = \frac{h}{V}\frac{\mathrm{d}}{\mathrm{d}t}[S], \tag{3.8}$$

where

$$h = \frac{(E^- -\rho_R V^2)[S]}{2\theta^- - G[S]}. \tag{3.9}$$

In view of the implication of (2.13), formula (2.9) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}[\eta] = V[\eta_X] = h\frac{\mathrm{d}}{\mathrm{d}t}[S], \tag{3.10}$$

and

$$[\eta_{XX}] = \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}[\eta_X]. \tag{3.11}$$

It is not difficult to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[S] = (V^2Ig^2+V^2g_S\text{-}g+V^2g_\eta\text{-}hg)[S_X]^2+Vg\frac{\mathrm{d}}{\mathrm{d}t}[S_X], \tag{3.12}$$

† Compare with Chen and Gurtin[1].

and

$$[\eta_{XX}] = (-hg^2I + h_S^- g^2 + h_\eta^- hg^2 + hIg^2 + hg_S^- g + h^2 g_\eta^- g)[S_X]^2 + \frac{1}{V} hg \frac{\mathrm{d}}{\mathrm{d}t}[S_X]. \quad (3.13)$$

Substituting the preceding results in (3.4) and (3.5) and combining the resulting relations, we obtain

$$\begin{aligned}\left(2\rho_R V - \rho_R V g + \frac{1}{V} G^- hg\right)\frac{\mathrm{d}}{\mathrm{d}t}[S_X] \\ = -(\hat{E}^- + \rho_R V^2 I g - \rho_R V^2 I g^2 - \rho_R V^2 g_S^- g - \rho_R V^2 h g_\eta^- g)[S_X]^2 \\ - G^-(h_S^- g^2 + h_\eta^- hg^2 + hg_S^- g + h^2 g_\eta^- g)[S_X]^2 \\ - 2\hat{T}^-_{S\eta} hg[S_X]^2 - \hat{T}^-_{\eta\eta} h^2 g^2 [S_X]^2 - (E^- - \rho_R V^2)[S_{XX}]. \quad (3.14)\end{aligned}$$

Formula (3.14) is the governing differential equation of induced discontinuities behind shock waves propagating in non-heat-conducting thermoelastic bodies. In the derivation of this equation it is assumed that the material bodies are initially at rest in their reference configurations. It is a simple matter to show that in the limit as the shock amplitudes go to zero and neglecting terms of $O([S])$, formula (3.14) reduces to

$$2\rho_R V \frac{\mathrm{d}}{\mathrm{d}t}[S_X] = -\hat{E}^+ [S_X]^2, \quad (3.15)$$

where $\rho_R V^2 = E^+$, and which is, of course, the governing differential equation of acceleration waves. The implications of (3.15) on the evolutionary behavior of acceleration waves are certainly well known[2].

The governing differential equations of the shocks (2.14) and of the induced discontinuities (3.14) may be written in more convenient forms. We note that

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = V \frac{\mathrm{d}}{\mathrm{d}X}[f],$$

so that (2.14) and (3.14) become

$$\frac{\mathrm{d}}{\mathrm{d}X}[S] = g[S_X], \quad (3.16)$$

$$\begin{aligned}(2\rho_R V^2 - \rho_R V^2 g + G^- hg)\frac{\mathrm{d}}{\mathrm{d}X}[S_X] \\ = -(\hat{E}^- + \rho_R V^2 I g - \rho_R V^2 I g^2 - \rho_R V^2 g_S^- g - \rho_R V^2 h g_\eta^- g)[S_X]^2 \\ - G^-(h_S^- g^2 + h_\eta^- hg^2 + hg_S^- g + h^2 g_\eta^- g)[S_X]^2 \\ - 2\hat{T}^-_{S\eta} hg[S_X]^2 - \hat{T}^-_{\eta\eta} h^2 g^2 [S_X]^2 - (E^- - \rho_R V^2)[S_{XX}]. \quad (3.17)\end{aligned}$$

Before we examine the implications of this coupled system of equations, let us recall certain classical results concerning shock transition.

4. CLASSICAL RESULTS OF SHOCK TRANSITION AND THEIR IMPLICATIONS

We first recall certain classical results of shock transition which are due to Bethe[3]. Given the conditions $(2.5)_{1,2}$ it is possible to show that balance of energy across the shock (2.12) implies:

(i) the shock must be compressive, i.e. $[S] < 0$;

(ii) the shock speed is supersonic with respect to the material ahead of the shock and subsonic with respect to the material behind the shock, i.e. $E^+ < \rho_R V^2 < E^-$; and

(iii) across the shock the jump in entropy increases with decreasing jump in strain.

It is also possible to show that in the limit as the shock amplitude becomes small the jump in entropy is proportional to third order of the jump in strain, viz.

$$[\eta] = \frac{1}{12}\frac{\tilde{E}^+}{\theta^+}[S]^3. \tag{4.1}$$

In general, there exists a function η_H giving the jump in entropy in terms of the jump in strain, i.e.

$$[\eta] = \eta_H([S]), \tag{4.2}$$

and in view of (3.10)

$$\frac{d}{dt}[\eta] = \frac{d\eta_H}{d[S]}\frac{d}{dt}[S] = h\frac{d}{dt}[S]. \tag{4.3}$$

Result (iii) implies that

$$\frac{d\eta_H}{d[S]} = h = \frac{(E^- - \rho_R V^2)[S]}{2\theta^- - G^-[S]} < 0, \tag{4.4}$$

so that

$$2\theta^- - G^-[S] > 0. \tag{4.5}$$

Given (4.5), it is a simple matter to show that the coefficient g, defined by (2.15), of the governing differential equation of the shock amplitudes is strictly negative

$$g < 0 \tag{4.6}$$

Thus, it follows from (3.16) that

$$\begin{aligned} [S_X] < 0 &\Leftrightarrow \frac{d}{dX}|[S]| < 0, \\ [S_X] > 0 &\Leftrightarrow \frac{d}{dX}|[S]| > 0. \end{aligned} \tag{4.7}$$

That is, whether a shock grows or decays depends on the sign of the induced discontinuity behind the shock. It is also a simple matter to show that the coefficient I, defined by (3.7) is strictly negative; and by (3.6) we have

$$\begin{aligned} \frac{d}{dX}|[S]| > 0 &\Leftrightarrow \frac{dV}{dt} > 0, \\ \frac{d}{dX}|[S]| < 0 &\Leftrightarrow \frac{dV}{dt} < 0. \end{aligned} \tag{4.8}$$

That is, the shock speed increases or decreases as the shock grows or decays. The results (4.7) and (4.8) are, of course, qualitatively consistent with those based on purely mechanical theory.

The coefficients g, h and I are, in general, functions of the jumps $[S]$ and $[\eta]$. In view of (4.2) we may define the function $\bar{g}$, $\bar{h}$ and $\bar{I}$ such that

$$\begin{aligned} g &= g([S], \eta_H([S])) = \bar{g}([S]), \\ h &= h([S], \eta_H([S])) = \bar{h}([S]), \\ I &= I([S], \eta_H([S])) = \bar{I}([S]). \end{aligned} \tag{4.9}$$

Given these functions, the governing differential equation (3.17) of induced discontinuities may be rewritten in a more convenient form. Specifically, we have

$$\frac{\mathrm{d}}{\mathrm{d}X}[S] = \bar{g}[S_X], \tag{4.10}$$

and

$$\begin{aligned} (2\rho_R V^2 - \rho_R V^2 \bar{g} + G^- \bar{h}\bar{g}) \frac{\mathrm{d}}{\mathrm{d}X}[S_X] = &-(\tilde{E}^- + \rho_R V^2 \bar{I}\bar{g} - \rho_R V^2 \bar{I}\bar{g}^2 - \rho_R V^2 \bar{g}_S \text{-} \bar{g})[S_X]^2 \\ &- G^-(\bar{h}_S \text{-} \bar{g}^2 + \bar{h}\bar{g}_S \text{-} \bar{g})[S_X]^2 - 2\hat{T}_{\bar{S}\eta}\bar{h}\bar{g}[S_X]^2 \\ &- \hat{T}_{\bar{\eta}\eta}\bar{h}^2\bar{g}^2[S_X]^2 - (E^- - \rho_R V^2)[S_{XX}]. \end{aligned} \tag{4.11}$$

Formulae (4.10) and (4.11) constitute a system of equations describing the coupled evolutionary behavior of shock amplitudes and induced discontinuities. In the following section, we shall consider approximations to these equations and examine their implications.

5. A SECOND ORDER APPROXIMATION

In our consideration of a second order approximation of the governing differential equations of the shock amplitudes and induced discontinuities we first presume that the constitutive relation for the internal energy is given by

$$e = e_0 + \tfrac{1}{2}E_0 S^2 + \tfrac{1}{6}\tilde{E}_0 S^3 + \theta_0 \eta + G_0 S\eta, \tag{5.1}$$

where $e_0, E_0, \tilde{E}_0, \theta_0$ and G_0 are constants. Given (5.1), it follows from (2.3) that the constitutive relations for the stress and the absolute temperature are

$$\begin{aligned} T &= E_0 S + \tfrac{1}{2}\tilde{E}_0 S^2 + G_0 \eta, \\ \theta &= \theta_0 + G_0 S. \end{aligned} \tag{5.2}$$

Given the constitutive relations (5.1) and $(5.2)_1$ it is a simple matter to show that balance of energy across the shock (2.12) again yields the result (4.1) with $\tilde{E}_0 = \tilde{E}^+$ and $\theta_0 = \theta^+$. It now follows from $(5.2)_1$ that the shock speed is given by

$$\rho_R V^2 = E_0 + \frac{1}{2}\tilde{E}_0[S] + \frac{1}{12}\frac{G_0 \tilde{E}_0}{\theta_0}[S]^2. \tag{5.3}$$

The presence of the third term in the right-hand member of (5.3) indicates that the shock speed is *greater* than that based on purely mechanical theory. Formula $(5.2)_1$ also implies that

$$E^- = E_0 + \tilde{E}_0[S] \tag{5.4}$$

so that

$$E^- - \rho_R V^2 = \frac{1}{2}\tilde{E}_0[S] - \frac{1}{12}\frac{G_0\tilde{E}_0}{\theta_0}[S]^2, \tag{5.5}$$

which is *less* than that based on purely mechanical theory. It is also necessary that

$$|[S]| < \frac{6\theta_0}{|G_0|}, \tag{5.6}$$

so that result (ii) of shock transition is satisfied. This places a purely *thermodynamic* restriction on the magnitude of the shock amplitude. It is of interest to note that (4.5) and $(5.2)_2$ imply $|[S]| > -2\theta_0/|G_0|$, which is, of course, satisfied trivially.

Given the preceding results it is not difficult to show that to within second order of the shock strength the coefficient $\bar{g}$ of the governing equation (4.10) reduces to

$$\bar{g} = -\frac{1}{4}\frac{\tilde{E}_0}{E_0}[S] + \frac{5}{32}\frac{\tilde{E}_0^2}{E_0^2}[S]^2 + \frac{1}{24}\frac{G_0\tilde{E}_0}{\theta_0 E_0}[S]^2. \tag{5.7}$$

Since $\bar{g}$ must be negative, we have two interesting conclusions. First, in view of the third term of the right-hand member of (5.7) the magnitude of $\bar{g}$ is less than that based on purely mechanical theory. Therefore, for the same value of the induced discontinuity $[S_X]$, the evolutionary behavior of the shock amplitude is *less* pronounced than that based on purely mechanical theory. Second, it is necessary that

$$|[S]| < \frac{1}{\dfrac{|G_0|}{6\theta_0} + \dfrac{5}{8}\dfrac{|\tilde{E}_0|}{E_0}} \tag{5.8}$$

in order that $\bar{g} < 0$ is satisfied. This places a *thermomechanical* restriction on the magnitude of the shock amplitude. We note that this restriction is stronger than the purely thermodynamic restriction (5.6). It is of interest to recall that the purely *mechanical* restriction on the magnitude of the shock amplitude is†

$$|[S]| < \frac{8}{5}\frac{E_0}{|\tilde{E}_0|}, \tag{5.9}$$

which is also weaker than the thermomechanical restriction.

To within second order of the shock strength the coefficients $\bar{I}$ and $\bar{h}$ are of the form

$$\bar{I} = \frac{1}{4}\frac{\tilde{E}_0}{E_0} - \frac{1}{8}\frac{\tilde{E}_0^2}{E_0^2}[S] + \frac{1}{12}\frac{G_0\tilde{E}_0}{\theta_0 E_0}[S] + \frac{1}{16}\frac{\tilde{E}_0^3}{E_0^3}[S]^2 - \frac{1}{16}\frac{G_0\tilde{E}_0^2}{\theta_0 E_0^2}[S]^2 - \frac{1}{12}\frac{G_0^2\tilde{E}_0}{\theta_0^2 E_0}[S]^2, \tag{5.10}$$

$$\bar{h} = \frac{1}{4}\frac{\tilde{E}_0}{\theta_0}[S]^2. \tag{5.11}$$

Further, it can be shown

$$2\rho_R V^2 - \rho_R V^2\bar{g} + G^-\bar{h}\bar{g} = 2E_0 + \frac{5}{4}\tilde{E}_0[S] - \frac{1}{32}\frac{\tilde{E}_0^2}{E_0}[S]^2 + \frac{1}{8}\frac{G_0\tilde{E}_0}{\theta_0}[S]^2, \tag{5.12}$$

† Compare with Bailey and Chen[4].

and

$$\tilde{E}^- + \rho_R V^2 I\bar{g} - \rho_R V^2 I\bar{g}^2 - \rho_R V^2 \bar{g}_S \cdot \bar{g} = \tilde{E}_0 - \frac{1}{8}\frac{\tilde{E}_0^2}{E_0}[S] + \frac{7}{64}\frac{\tilde{E}_0^3}{E_0^2}[S]^2 + \frac{1}{48}\frac{G_0\tilde{E}_0^2}{\theta_0 E_0}[S]^2. \quad (5.13)$$

In addition,

$$\bar{h}_S \cdot \bar{g}^2 = O([S]^3),$$
$$\bar{h}\bar{g}_S \cdot \bar{g} = O([S]^3),$$
$$\bar{h}^2\bar{g}^2 = O([S]^6).$$

The governing differential equation of induced discontinuities to within second order of the shock strength may now be derived. Indeed, substituting the preceding results into (4.11) we obtain upon further reduction the result

$$\frac{\mathrm{d}}{\mathrm{d}X}[S_X] = \left\{-\frac{1}{2}\frac{\tilde{E}_0}{E_0} + \frac{3}{8}\frac{\tilde{E}_0^2}{E_0^2}[S] - \frac{19}{64}\frac{\tilde{E}_0^3}{E_0^3}[S]^2 + \frac{1}{48}\frac{G_0\tilde{E}_0^2}{\theta_0 E_0^2}[S]^2\right\}[S_X]^2$$
$$+\left\{-\frac{1}{4}\frac{\tilde{E}_0}{E_0}[S] + \frac{5}{32}\frac{\tilde{E}_0^2}{E_0^2}[S]^2 + \frac{1}{24}\frac{G_0\tilde{E}_0}{\theta_0 E_0}[S]^2\right\}[S_{XX}]. \quad (5.14)$$

The implications of (5.14) depend, of course, on the nature of its coefficients. The coefficient of the $[S_{XX}]$ term is $\bar{g}$, given by (5.7). Therefore, the influence of $[S_{XX}]$ is *less* pronounced than that due to purely mechanical theory. The properties of the coefficient of the $[S_X]^2$ term are more difficult to ascertain. First, this coefficient does not vanish for any negative value of $[S]$ which satisfies the thermomechanical restriction (5.8). Second, the values of this coefficient are positive for all negative values of $[S]$ satisfying (5.8). The presence of its fourth term thus implies that its magnitude is less than that based on purely mechanical theory. Therefore, for the same value of $[S_X]$ and with $[S_{XX}] = 0$ the evolutionary behavior of induced discontinuities is less pronounced than that based on purely mechanical theory. This situation is similar to that for the evolutionary behavior of shock amplitudes.

In closing, let us examine certain features of the solution of the reduced equations of shock amplitudes and induced discontinuities, viz.

$$\frac{\mathrm{d}}{\mathrm{d}X}[S] = (\alpha[S] + \beta[S]^2)[S_X], \qquad \frac{\mathrm{d}}{\mathrm{d}X}[S_X] = (2\alpha + \gamma[S] + \delta[S]^2)[S_X]^2, \quad (5.15)$$

where we have neglected the term involving $[S_{XX}]$ whose magnitude is presumed to be small, and where

$$\alpha = -\frac{1}{4}\frac{\tilde{E}_0}{E_0}, \qquad \beta = \frac{5}{32}\frac{\tilde{E}_0^2}{E_0^2} + \frac{1}{24}\frac{G_0\tilde{E}_0}{\theta_0 E_0},$$
$$\gamma = \frac{3}{8}\frac{\tilde{E}_0^2}{E_0^2}, \qquad \delta = -\frac{19}{64}\frac{\tilde{E}_0^3}{E_0^3} + \frac{1}{48}\frac{G_0\tilde{E}_0^2}{\theta_0 E_0^2}. \quad (5.16)$$

Rewriting $(5.15)_1$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}X}|[S]| = (\alpha|[S]| - \beta|[S]|^2)[S_X], \quad (5.17)$$

and if $[S_X]$ is regarded to be a function of $|[S]|$, then $(5.15)_2$ and (5.17) imply

$$\frac{d}{d|[S]|}[S_X] = \frac{2\alpha - \gamma|[S]| + \delta|[S]|^2}{\alpha|[S]| - \beta|[S]|^2}[S_X]. \tag{5.18}$$

The solution of (5.18) for $\alpha - \beta|[S]| > 0$ is of the form

$$[S_X] = C|[S]|^2(\alpha - \beta|[S]|)^{(\gamma/\beta - \alpha\delta/\beta^2 - 2)} \exp\left(-\frac{\delta}{\beta}|[S]|\right), \tag{5.19}$$

where C is a constant given by the initial value of $[S_X]$ and $\operatorname{sgn} C = \operatorname{sgn}[S_X]$. The condition $\alpha - \beta|[S]| > 0$ is certainly consistent with the requirement $\bar{g} < 0$. Further, $(5.15)_2$ implies that if $[S_X]$ is zero at some X, then it is always zero. Therefore, the solution (5.19) is valid for these values of $|[S]|$ which satisfy the restriction

$$0 < |[S]| < \frac{1}{\dfrac{|G_0|}{6\theta_0} + \dfrac{5}{8}\dfrac{|\bar{E}_0|}{E_0}}. \tag{5.20}$$

Since

$$\frac{\gamma}{\beta} - \frac{\alpha\delta}{\beta^2} - 2 < 0,$$

and since $2\alpha + \gamma[S] + \delta[S]^2$ does not vanish in the interval specified by (5.20), it follows from (5.19) that $|[S_X]|$ is a monotonically increasing function of $|[S]|$ within the interval specified by (5.20) such that

$$|[S_X]| \to 0 \qquad \text{as } |[S]| \to 0,$$

and

$$|[S_X]| \to \infty \qquad \text{as } |[S]| \to \frac{1}{\dfrac{|G_0|}{6\theta_0} + \dfrac{5}{8}\dfrac{|\bar{E}_0|}{E_0}}.$$

The implications of these results are interesting. First, we note that if $[S_X] < 0$, then both $|[S_X]|$ and $|[S]|$ will decrease to zero. On the other hand, if $[S_X] > 0$ then both $[S_X]$ and $|[S]|$ will increase such that

$$[S_X] \to \infty \quad \text{as } |[S]| \to \frac{1}{\dfrac{|G_0|}{6\theta_0} + \dfrac{5}{8}\dfrac{|\bar{E}_0|}{E_0}}.$$

Given the solution (5.19) of $[S_X]$ in terms of $|[S]|$, the governing differential equation $(5.15)_1$ of shock amplitudes becomes

$$\frac{d}{dX}|[S]| = C|[S]|^3(\alpha - \beta|[S]|)^{(\gamma/\beta - \alpha\delta/\beta^2 - 1)} \exp\left(-\frac{\delta}{\beta}|[S]|\right). \tag{5.21}$$

For sufficiently small propagation distances a series solution of the form

$$|[S](X)| = |[S](0)| + a_1 X + a_2 X^2 + \dots, \tag{5.22}$$

may be obtained for (5.21) with

$$a_1 = C|[S](0)|^3(\alpha-\beta|[S](0)|)^{(\gamma/\beta-\alpha\delta/\beta^2-1)}\exp\left(-\frac{\delta}{\beta}|[S](0)|\right), \tag{5.23}$$

$$a_2 = \frac{1}{2}C|[S](0)|^3(\alpha-\beta|[S](0)|)^{(\gamma/\beta-\alpha\delta/\beta^2-1)}\exp\left(-\frac{\delta}{\beta}|[S](0)|\right)$$

$$\times\left(\frac{3}{|[S](0)|}-\frac{(\gamma/\beta-\alpha\delta/\beta^2-1)\beta}{\alpha-\beta|[S](0)|}-\frac{\delta}{\beta}\right)a_1. \tag{5.24}$$

These results should prove quite useful in the reduction of experimental data.

Finally, we note that there is no technical difficulty in including the term involving $[S_{XX}]$ in $(5.15)_2$. We simply obtain additional terms in the solution of $[S_X]$ which are proportional to the least upper bound of $|[S_{XX}]|$. As long as the least upper bound remains small, the contributions of these additional terms are negligible in their influences on the evolutionary behavior of the shock amplitudes and the induced discontinuities under consideration.

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